More about Vacuum Spacetimes with Toroidal Null Infinities

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Recently Bernd Schmidt has given three explicit examples of spacetimes with toroidal null infinities. In this paper all solutions with a toroidal null infinity within Schmidt's metric ansatz (polarized Gowdy models) are constructed. The members of the family are determined by two smooth functions of one variable. For the unpolarized Gowdy models the same kind of analysis carries through.

I. INTRODUCTION

In [1] Bernd Schmidt gave three examples of spacetimes which can be mapped to an unphysical[1] spacetime with regular boundary with the topology "torus times line". Those examples fulfill the conditions regarding regularity in the definition of asymptotic simplicity, but not the condition that every null geodesic starts at past null infinity and ends at future null infinity[2]. As the metric A3 in the classification of Ehlers and Kundt is the simplest known solution with this structure of null infinity I will call those solutions **asymptotically A3**.

If one solves an initial value problem numerically in physical spacetime, the topological differences of asymptotically A3 solutions to asymptotically flat solutions do not show up in a essential way. The grid has periodic boundary conditions with respect to two spacelike coordinates, and "normal" boundary conditions for the remaining spacelike coordinate. Gravitational radiation in asymptotically A3 spacetimes is emitted through the boundaries limiting the range of the latter coordinate. Calculating the gravitational radiation emitted is pure algebra (see equation (9) and [4]). Therefore these solutions may not only serve as testbeds for the numerical calculation of solutions from initial data (especially for the outer boundary treatment) but they may also serve as testbed for the radiation extraction procedures.

The explicit solutions given by Bernd Schmidt originate in the solution of a 1+1 wave equation. For any smooth data (two functions of one variable) one gets a smooth solution of Einstein's equation, but not necessarily one which is asymptotically A3. He conjectured

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[1]See e. g. [2, Definition 1 and the following text] for an explanation of the distinction between physical and unphysical spacetimes in the context of asymptotical flatness.

[2]With this condition the conformal boundary must have the topology "sphere times line" [3]

that every solution of the 1+1 wave equation which can be extended smoothly to null infinity yields an asymptotically A3 solution, i. e. the other function determining the metric necessarily has a smooth limit at null infinity also.
In this article I give necessary and sufficient conditions for asymptotically A3 solutions on the initial data of the wave equation(s) for polarized and unpolarized Gowdy spacetimes. For polarized Gowdy models further explicit solutions are guessed. Generally they radiate away part of their Bondi mass.

II. CONSTRUCTION OF THE SOLUTIONS

As described in [1][3], for the metric

$$\tilde{g} = \frac{1}{\sqrt{\tilde{t}}}\, e^{\tilde{N}}(-d\tilde{t}^2 + d\tilde{z}^2) + \tilde{t}\,(e^{\tilde{W}} d\tilde{x}^2 + e^{-\tilde{W}} d\tilde{y}^2) \tag{1}$$

the field equations are

$$\tilde{W}_{,\tilde{t}\tilde{t}} + \frac{1}{\tilde{t}}\,\tilde{W}_{\tilde{t}} - \tilde{W}_{,\tilde{z}\tilde{z}} = 0, \tag{2a}$$

$$\tilde{N}_{,\tilde{t}} - \frac{\tilde{t}}{2}\left(\left(\tilde{W}_{,\tilde{t}}\right)^2 + \left(\tilde{W}_{,\tilde{z}}\right)^2\right) = 0, \tag{2b}$$

and

$$\tilde{N}_{,\tilde{z}} - \tilde{W}_{,\tilde{t}}\,\tilde{W}_{,\tilde{z}} = 0. \tag{2c}$$

$\tilde{W}$ and $\tilde{N}$ are functions of $\tilde{t}$ and $\tilde{z}$. The $\tilde{}$ flags quantities which are supposed to live in physical spacetime in contrast to objects without $\tilde{}$ which live in unphysical spacetime. The wave equation (2a) is the integrability condition of the equations (2b) and (2c). Therefore, giving smooth data $\tilde{W}$ and $\tilde{W}_{,\tilde{t}}$ on any spacelike surface with $\tilde{t} = \tilde{t}_0$, one obtains smooth functions $\tilde{W}(\tilde{t},\tilde{z})$ and $\tilde{N}(\tilde{t},\tilde{z})$ for all $\tilde{t} > 0$. But those are not, depending on the data, necessarily asymptotically A3.
To show that such a spacetime $(\tilde{M},\tilde{g})$ is indeed asymptotically A3 one has to find a conformal mapping Ω with $g = \Omega^2\tilde{g}$ and an unphysical spacetime (M, g) with smooth boundary $\mathcal{J}$. To find all asymptotically A3 solutions it is advantageous to proceed in the other direction, i. e. to find equations for the functions W and N in the unphysical spacetime which guarantee that (2) is fulfilled for the corresponding $\tilde{W}$ and $\tilde{N}$ and which ensure that for smooth initial data for W and N on M the functions W and N are smooth in a sufficient region of M. The functions $\tilde{W}$ and $\tilde{N}$ on physical spacetime are related to W and N on unphysical spacetime by $\tilde{W}(\tilde{t},\tilde{z}) = W(t(\tilde{t},\tilde{z}), z(\tilde{t},\tilde{z}))$ and $\tilde{N}(\tilde{t},\tilde{z}) = N(t(\tilde{t},\tilde{z}), z(\tilde{t},\tilde{z}))$.
As in [1] we map the coordinates $(\tilde{t},\tilde{z},\tilde{x},\tilde{y})$ to (t, z, x, y) by

$$\tilde{t} = 8\,\frac{t^2 + z^2}{(t^2 - z^2)^2}, \tag{3a}$$

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[3]Note that the metric is written in terms of the functions $\tilde{W}$ and $\tilde{M} = -1/4\,\ln\tilde{t} + \tilde{N}/2$ in reference [1].

$$\tilde{z} = 16\,\frac{t\,z}{\left(t^2 - z^2\right)^2}, \tag{3b}$$

$$\tilde{x} = x, \tag{3c}$$

$$\tilde{y} = y. \tag{3d}$$

As we want to get all asymptotically A3 solutions within the metric ansatz (1), we have to know whether the coordinate transformation (3) for the "compactification" is essentially unique. As there are two independent, non-vanishing, hypersurface-orthogonal spacelike Killing vector fields the choice of null directions in the two dimensional subspace orthogonal to the Killing orbits is unique up to a null boost and those null directions cover the whole globally hyperbolic part of the spacetime. The "compactification" $u = f(\tilde{u})$ and $v = g(\tilde{v})$ must lead to a regular and smooth unphysical metric which fixes the "compactification" up to a factor. Transformation (3) is this essentially unique "coordinate compactification".
The transformation (3) maps $(\tilde{t}, \tilde{z}) \in (\,]0,\infty[\,,\,]-t,t[\,)$ to $(t,z) \in (\,]\infty,0[\,,\,]-t,t[\,)$. The notation $]\infty,0[$ emphases that an increasing $\tilde{t}$ corresponds to a decreasing t. The inverse of transformation (3) is

$$t = \sqrt{\frac{2}{\tilde{t}-\tilde{z}}} + \sqrt{\frac{2}{\tilde{t}+\tilde{z}}}, \tag{4a}$$

$$z = \sqrt{\frac{2}{\tilde{t}-\tilde{z}}} - \sqrt{\frac{2}{\tilde{t}+\tilde{z}}}. \tag{4b}$$

With the definitions

$$\Omega = \frac{1}{4}\left(t^2 - z^2\right), \tag{5a}$$

and

$$g = \frac{4\sqrt{2}}{\sqrt{t^2+z^2}}\, e^N \left(-dt^2 + dz^2\right) + \frac{1}{2}\left(t^2+z^2\right)\left(e^W dx^2 + e^{-W} dy^2\right) \tag{5b}$$

we have $g = \Omega^2\,\tilde{g}$. For W and N smooth on $(t,z) = (\,]0,\infty[\,,\,[-t,t]\,)$ the spacetime $(\tilde{M},\tilde{g})$ is by definition asymptotically A3.
The equations (2) become

$$\left(t^4 - z^4\right)\left(W_{,tt} - W_{,zz}\right) - 2\,t\left(3\,z^2 + t^2\right) W_{,t} - 2\,z\left(z^2 + 3\,t^2\right) W_{,z} = 0, \tag{6a}$$

$$N_{,t} + \frac{t^2+z^2}{4\left(t^2-z^2\right)^2}\left(\,t\,\left(3\,z^2+t^2\right)\left(\left(W_{,t}\right)^2 + \left(W_{,z}\right)^2\right) + 2\,z\,\left(z^2+3\,t^2\right)\,W_{,t}\,W_{,z}\,\right) = 0, \tag{6b}$$

and

$$N_{,z} + \frac{t^2+z^2}{4\left(t^2-z^2\right)^2}\left(\,z\,\left(z^2+3\,t^2\right)\left(\left(W_{,t}\right)^2 + \left(W_{,z}\right)^2\right) + 2\,t\,\left(3\,z^2+t^2\right)\,W_{,t}\,W_{,z}\,\right) = 0. \tag{6c}$$

These equations are singular on $\mathscr{J}$ where $t^2 - z^2 = 0$. Therefore giving smooth data W and $W_{,t}$ on some slice $(t_0, [-t_0, t_0])$ one does not know a priori whether the solution W remains smooth. But for $W(t,z)$ smooth it follows from equation (6a) that at $t = z$ the directional derivative $W_{,t} + W_{,z} = 0$ and at $t = -z$ the directional derivative $W_{,t} - W_{,z} = 0$. Therefore any smooth solution W of (6a) necessarily has the form

$$W(t,z) = f(t,z)(t^2 - z^2) + \mathrm{const} \tag{7}$$

with smooth f. By a rescaling of x and y one can always achieve const $= 0$. The equations (6) become

$$(f_{,tt} - f_{,zz}) - \frac{2\,z}{t^2+z^2} f_{,z} + \frac{2\,t}{t^2+z^2} f_{,t} = 0, \tag{8a}$$

$$N_{,t} + \frac{t^2+z^2}{4}\Bigg(4\,t\,(f + 2\,z\,f_{,z})\,f + t\left(3\,z^2+t^2\right)(f_{,z})^2 + 4\,f\left(t^2+z^2\right) f_{,t} + 2\,z\left(z^2+3\,t^2\right) f_{,t}\,f_{,z} + t\left(3z^2+t^2\right)(f_{,t})^2 \Bigg) = 0, \tag{8b}$$

and

$$N_{,z} + \frac{t^2+z^2}{4}\Bigg(4\,z\,(f + 2\,t\,f_{,t})\,f + z\left(z^2+3\,t^2\right)(f_{,z})^2 + 4\,f\left(t^2+z^2\right) f_{,z} + 2\,t\left(3\,z^2+t^2\right) f_{,t}\,f_{,z} + z\left(z^2+3\,t^2\right)(f_{,t})^2 \Bigg) = 0, \tag{8c}$$

which is regular for $t > 0$.
If we give smooth data f and $f_{,t}$ on $(t_0,]-\infty,\infty[)$ equation (8a) can be solved on $(]0,\infty[,]-\infty,\infty[)$. The solution $f(t,z)$ is smooth. Therefore integration of (8b) and (8c) gives a smooth $N(t,z)$. Hence we have asymptotically A3 solutions of Einstein's equation.
As the solutions in general do not have a timelike Killing vector field, as do spherical symmetric vacuum solutions they are candidates for exact solutions with gravitational radiation. The expression for the Bondi mass written in terms of unphysical variables evaluated at the conformal boundary is

$$m_{\mathrm{Bondi}} = c \cdot e^{-N}\Big|_{\mathcal{J}}, \tag{9}$$

with a positive constant c depending on the period of x and y [4]. The derivatives along the null infinities are decreasing, which can be seen by using equation (8b) and (8c).

Two everywhere regular solutions of (8a) can easily be guessed. The first, $f(t,z) = c$, corresponds to

$$\tilde{W}(\tilde{t},\tilde{z}) = \frac{8\,c}{\sqrt{\tilde{t}^2 - \tilde{z}^2}}, \tag{10a}$$

$$\tilde{N}(\tilde{t},\tilde{z}) = \frac{-16\,c^2\,\tilde{t}^2}{\left(\tilde{t}^2 - \tilde{z}^2\right)^2}, \tag{10b}$$

which is up to the constant c the solution (3.6) of [1].
The second, $f(t,z) = c\,t\,z$, corresponds to

$$\tilde{W}(\tilde{t},\tilde{z}) = \frac{32\,c\,\tilde{z}}{\sqrt{\tilde{t}^2 - \tilde{z}^2}^{\,3}}, \tag{11a}$$

$$\tilde{N}(\tilde{t},\tilde{z}) = \frac{-128\,c^2\,\tilde{t}^2\left(8\tilde{z}^2 + \tilde{t}^2\right)}{\left(\tilde{t}^2 - \tilde{z}^2\right)^4}. \tag{11b}$$

For general Gowdy models [5],

$$\tilde{g} = \frac{1}{\sqrt{\tilde{t}}}\, e^{\tilde{N}}(-d\tilde{t}^2 + d\tilde{z}^2) + \tilde{t}\, \left(e^{\tilde{W}}\, d\tilde{x}^2 + 2e^{\tilde{W}}\, d\tilde{x}\, d\tilde{y} + (e^{\tilde{W}}\tilde{Q}^2 + e^{-\tilde{W}})\, d\tilde{y}^2\right), \tag{12}$$

an equivalent procedure carries through. It is again necessary and sufficient for asymptotically A3 that both of the functions $\tilde{W}$ and $\tilde{Q}$, which satisfy a system of nonlinear, coupled wave equations, fall off such that

$$W(t,z) = f(t,z)(t^2 - z^2) + \text{const} \tag{13}$$

and

$$Q(t,z) = g(t,z)(t^2 - z^2) + \text{const.} \tag{14}$$

The wave equations become regular equations for f and g and the equations for N are also regular. But due to the nonlinear coupling of the wave equations, guessing solutions which are not polarized ($Q = 0$), may be difficult.

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